Int. J. Solids Structures, 1972, Vol. 8, pp. 385 to 399. Pergamon Press. Printed in Great Britain

ON THE EQUATIONS FOR COMPOSITE BEAMS UNDER INITIAL STRESS

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Abstract—The composite beam consisting of a large number of alternating layers of two different elastic materials under initial stress is considered. The constituent layers are assumed to behave as Timoshenko beams under initial stress. With the aid of a smoothing operation, the composite beam is transformed into a macrohomogeneous beam with micro-structure. The equations of motion as well as the boundary conditions are derived by Hamilton's principle. A set of approximate equations is derived based in part on the distinctly differential rigidities of the two constituent materials. Flexural wave propagation and the stability of a composite beam are investigated by using the micro-structure beam theory, the approximate theory and the law of mixtures. Numerical examples are given.

1. INTRODUCTION

IN THE analysis of mechanical responses of a composite material it is often formidable to treat the constituents individually. To make the analysis manageable, it is a common practice to regard the composite as a grossly homogeneous medium, for which the classical continuum theories apply. However, such a drastic approximation has limitations in applications, especially in dynamic cases where the wavelength of deformation may become comparable to the characteristic length of inhomogeneity of the composite. The difficulty can be resolved to a great extent if microstructure is introduced in the continuum model to account for the intrinsic motion in the constituents.

In a recent paper [1], the author proposed a micro-structure theory for a laminated composite beam. A set of equations of motion was derived, which was employed to study harmonic flexural wave propagation in the composite beam. It was found that the proposed theory yielded results which were in very good agreement with those obtained according to an exact analysis. It was also observed that continuum models without micro-structure were adequate only in the range of very long wavelengths.

Encouraged by the result of [1], the micro-structure beam theory is now extended in this paper to incorporate the initial stress.

Strictly speaking, initial stresses are always present in composite materials. Since almost all the composites are manufactured at a temperature different from the service temperature, interactions caused by the different thermal expansion (or contraction) between constituents are usually inevitable. Moreover, many composites are designed to make reinforcing fibers (or layers) as prestressing devices. Initial stress induced by externally applied mechanical means have, of course, long been of great interest.

In this paper, for the sake of simplicity but without losing practical significance, only the initial stresses applied in the axial direction are considered. Equations of motion and boundary conditions are derived by a variational principle. The equations thus obtained are used to investigate the influence of initial stress on the flexural wave propagation in the composite beam. A subsequent investigation in the stability of the vibration modes leads to the buckling stresses. A set of approximate equations is also presented and discussed. Results according to the law of mixtures are also included; and comparison with the micro-structure theory is made.

2. INCREMENTAL ENERGIES

Consider a composite beam which is composed of a large number of periodically alternating plane layers of two different homogeneous elastic materials. For the sake of brevity, the cross-section of the composite beam is assumed to be rectangular with the depth h and the width b. The thicknesses of the layers of materials 1 and 2 are denoted by d_1 and d_2, respectively, see Fig. 1. With no loss of generality, we assume that material 1 is stiffer than material 2.

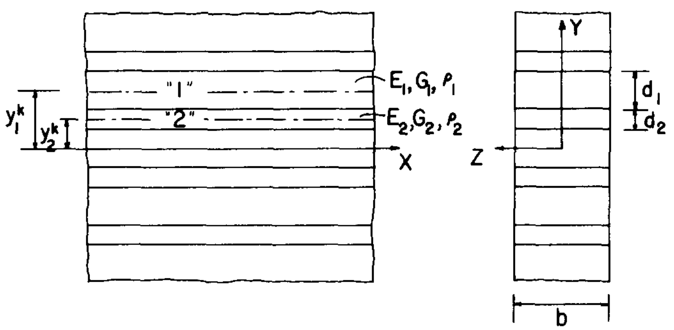

FIG. 1. Composite beam.

The beam is initially straight and subjected to initial stresses applied along the axis of the beam. Let x, y be the coordinates of a particle of the body in the initial state, and x-axis coincide with the centroidal axis of the beam. The initial stresses in the layers of material 1 and material 2 are denoted by σ_1^0 and σ_2^0, respectively. It is assumed that σ_1^0 and σ_2^0 are constants and, in general, of different values.

We now assume that each layer behaves as a Timoshenko beam under initial stress. Two sets of governing equations for a homogeneous Timoshenko beam under initial stress have been derived in [2] based on two three-dimensional theories for incremental deformations as proposed by Trefftz [3] and Biot [4], respectively. In this paper, the formulation according to Trefftz's theory will be taken.

Consider a typical constituent layer, say the **k**th layer. The incremental motion in the layer consists of an extension at the mid-plane, u^k, a rotation of the cross-section, ϕ^k, a shear deformation, β^k, and a transverse deflection w^k, in conjunction with the relation

$$\phi^k + \beta^k = \frac{\partial w^k}{\partial x}. \tag{1}$$

We assume that the gross behavior of the composite beam also obeys the Timoshenko beam theory. If a state of flexural deformation is assumed, then two kinematic variables are needed to describe the gross deformation of the beam, namely, the transverse deflection of the neutral axis, w, and the gross rotation ψ. It is important to note that, since the constituent layers differ in material property, the composite beam does not possess a plane cross-section after deformation as the beam of a homogeneous medium. The gross rotation ψ should then be interpreted as a fictitious plane on which lie the average longitudinal displacements of the constituent layers, u^k. With the foregoing in mind, we can write

$$u_i^k = -y_i^k \psi(x, t) \qquad i = 1, 2 \tag{2}$$

for the **k**th stiff reinforcing layer ($i = 1$) and the **k**th matrix layer ($i = 2$), respectively. In equation (2), y_1^k and y_2^k denote the coordinates of the centroidal axes of the **k**th reinforcing and matrix layers, respectively, see Fig. 1.

Since we are interested in the incremental deformations that are flexural in nature, we may further assume that the slopes of deflection in all the layers are the same and equal to that of the composite beam, i.e.

$$\frac{\partial w_i^k}{\partial x} = \frac{\partial w}{\partial x}, \qquad i = 1, 2. \tag{3}$$

In view of equation (3), the relation given by equation (1) can be written as

$$\phi_i^k(x, t) + \beta_i^k(x, t) = \frac{\partial w(x, t)}{\partial x}, \qquad i = 1, 2 \tag{4}$$

for the **k**th pair of layers. From equation (4), we note that the local rotations ϕ_i^k and the local shear deformations β_i^k in the constituent layers are now functions of x and time t.

The assumption regarding the deformation in the composite beam as given by equations (2) and (4) has been employed to derive a set of equations of motion for a composite beam without initial stress [1]. The equations obtained in [1] have been shown to be very accurate in predicting the frequencies of flexural vibrations of the composite beam [1].

Using Trefftz's incremental stress components which are assumed to be related linearly to the accompanying deformation, we can obtain an expression for the incremental strain energy for an initially stressed homogeneous Timoshenko beam in bending, shear deformation and extension. The derivation is presented in the Appendix. The incremental strain energy per unit length in the **k**th reinforcing layer and the **k**th matrix layer are, see Appendix,

$$\Delta U_i^k = \frac{1}{2}E_i I_i \left(\frac{\partial \phi_i^k}{\partial x}\right)^2 + \frac{1}{2}\kappa_i A_i G_i (\beta_i^k)^2 + \frac{1}{2}E_i A_i (y_i^k)^2 \left(\frac{\partial \psi}{\partial x}\right)^2$$
$$+ \frac{1}{2}\beta A_i \sigma_i^0 (y_i^k)^2 \left(\frac{\partial \psi}{\partial x}\right)^2 + \frac{1}{2}\beta I_i \sigma_i^0 \left(\frac{\partial \phi_i^k}{\partial x}\right)^2 + \frac{1}{2}A_i \sigma_i^0 \left(\frac{\partial w}{\partial x}\right)^2 \qquad i = 1, 2 \tag{5}$$

where A_i, I_i, κ_i, E_i and G_i are, respectively, the cross-sectional area, the moment of inertia, the shear correction coefficient, the Young's modulus and the shear modulus of the respective layer. The parameter $\beta(=1, 0)$ is introduced to attach to two terms in equation (5), which will be shown to be negligible when the initial stresses are small as compared with the elastic moduli. The elastic moduli in equation (5) depend, in general, on the initial stress. Thus, each layer may possess transverse isotropy due to either intrinsic material property or the presence of initial stress. The value of the shear correction coefficient for a Timoshenko beam under initial stress was discussed in [2], and was found to be $\pi^2/12$ if Trefftz's formulation is taken.

The kinetic energies per unit length of the **k**th reinforcing and matrix layers are obtained as

$$T_i^k = \frac{1}{2}\rho_i A_i\left(\frac{\partial w}{\partial t}\right)^2 + \frac{1}{2}\rho_i I_i\left(\frac{\partial \phi_i^k}{\partial t}\right)^2 + \frac{1}{2}\rho_i A_i (y_i^k)^2\left(\frac{\partial \psi}{\partial t}\right)^2, \qquad i = 1, 2. \tag{6}$$

In deriving equation (6), the kinetic energies in the individual layers due to the thickness–stretch motion are neglected.

We assume that the layers are perfectly bonded. The continuity of displacement at the interface of the **k**th reinforcing and matrix layers is ensured by the equation

$$\psi = \eta\phi_1^k + (1-\eta)\phi_2^k \tag{7}$$

where

$$\eta = d_1/(d_1+d_2). \tag{8}$$

In view of equations (2), (4) and (7), we can eliminate β_i^k and ϕ_2^k from equations (5) and (6), and keep w, ψ and ϕ_1^k as independent kinematic variables. It is noted that ϕ_1^k is a "discrete" variable, since it is defined only at the mid-plane of the **k**th reinforcing layer. Basic to the continuum theory is the use of kinematic variables which are continuous functions of the spatial domain. A smoothing operation was employed in [1] to convert the discrete variables into continuous variables, thus transforming the discrete system of layers in the composite beam into a macrohomogeneous beam with micro-structure. The basic scheme involved in the smoothing operation for the present problem is exhibited by the following evaluation of the incremental energies for the composite beam:

$$\Delta U = \sum_k (\Delta U_1^k + \Delta U_2^k) \simeq \int_{-h/2}^{h/2} \frac{1}{d_1+d_2}(\Delta U_1 + \Delta U_2)\,\mathrm{d}y \tag{9}$$

$$T = \sum_k (T_1^k + T_2^k) \simeq \int_{-h/2}^{h/2} \frac{1}{d_1+d_2}(T_1 + T_2)\,\mathrm{d}y \tag{10}$$

where ΔU is the incremental strain energy and T the kinetic energy per unit length of the composite beam. The approximation by using a weighted integral for a summation as has been done in equations (9) and (10) requires that ϕ_1^k be a function of y rather than y_i^k. To distinguish the continuous variable from the discrete variable we henceforth remove the superscript k to indicate the corresponding continuous quantity.

After integration, equations (9) and (10) yield the expressions for the incremental strain and kinetic energies in terms of the "smoothed" kinematic variables as

$$
\begin{aligned}
\Delta U = {} & \frac{\xi}{2}(E_1 I_1 + \beta I_1 \sigma_1^0)\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\kappa_1 A_1 G_1 \left(\frac{\partial w}{\partial x} - \phi\right)^2 \\
& + \frac{\xi}{2}(E_2 I_2 + \beta I_2 \sigma_2^0)\left(\frac{1}{1-\eta}\frac{\partial \psi}{\partial x} - \frac{1}{1-\eta}\frac{\partial \phi}{\partial x}\right)^2 \\
& + \frac{1}{2}\kappa_2 \xi A_2 G_2 \left(\frac{\partial w}{\partial x} - \frac{1}{1-\eta}\psi + \frac{1}{1-\eta}\phi\right)^2 \\
& + \frac{1}{2} I_b[\eta E_1 + (1-\eta)E_2 + \beta\eta\sigma_1^0 + \beta(1-\eta)\sigma_2^0]\left(\frac{\partial \psi}{\partial x}\right)^2 \\
& + \frac{1}{2}\xi(A_1\sigma_1^0 + A_2\sigma_2^0)\left(\frac{\partial w}{\partial x}\right)^2
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
T = {} & \frac{1}{2}\xi(\rho_1 A_1 + \rho_2 A_2)\left(\frac{\partial w}{\partial t}\right)^2 + \frac{1}{2}\xi\rho_1 I_1 \left(\frac{\partial \phi}{\partial t}\right)^2 \\
& + \frac{1}{2}\xi\rho_2 I_2 \left(\frac{1}{1-\eta}\frac{\partial \psi}{\partial t} - \frac{\eta}{1-\eta}\frac{\partial \phi}{\partial t}\right)^2 \\
& + \frac{1}{2} I_b[\eta\rho_1 + (1-\eta)\rho_2]\left(\frac{\partial \psi}{\partial t}\right)^2
\end{aligned} \tag{12}
$$

where $\phi \equiv \phi_1$, I_b is the gross sectional moment of inertia of the composite beam, and ξ is defined as

$$\xi = h/(d_1 + d_2). \tag{13}$$

3. EQUATIONS OF MOTION AND BOUNDARY CONDITIONS

Hamilton's principle is now applied to derive the equations of motion. The principle states that for an arbitrary time interval t_0 to t_1

$$\delta \int_{t_0}^{t_1} \int_0^l (T - \Delta U)\, \mathrm{d}x\, \mathrm{d}t + \int_{t_0}^{t_1} \delta W_e\, \mathrm{d}t = 0 \tag{14}$$

for arbitrary values of δw, $\delta\psi$ and $\delta\phi$. In equation (14), l is the length of beam and δW_e is the variation of work done by the additional external forces which induce the incremental deformation. The expression of δW_e is the same as the variation of work done by external forces for a composite beam without initial stress given by [1]. We have

$$\delta W_e = \int_0^l q\delta w\, \mathrm{d}x + [\overline{M}\delta\psi]_0^l + [Q\delta W]_0^l + [\overline{m}\delta\phi]_0^l \tag{15}$$

where $q(x, t)$ is the lateral load per unit initial span acting in the direction of the y-axis; $\overline{M}$ is interpreted as the incremental gross moment, Q the incremental shear force and $\overline{m}$ the incremental micro-moment which acts on the individual layers.

The Euler equations for the variational equation, equation (14), are the desired equations of motion. After substituting equations (11), (12) and (15) into equation (14), a system of displacement-equations of motion can be obtained as

$$b_1\frac{\partial^2 w}{\partial x^2}-b_2\frac{\partial \psi}{\partial x}-b_3\frac{\partial \phi}{\partial x}+q/\xi = b_4\frac{\partial^2 w}{\partial t^2} \tag{16}$$

$$b_2\frac{\partial w}{\partial x}+b_5\frac{\partial^2 \psi}{\partial x^2}-b_6\psi-b_7\frac{\partial^2 \phi}{\partial x^2}+b_8\phi = b_9\frac{\partial^2 \psi}{\partial t^2}-b_{10}\frac{\partial^2 \phi}{\partial t^2} \tag{17}$$

$$b_3\frac{\partial w}{\partial x}-b_7\frac{\partial^2 \psi}{\partial x^2}+b_8\psi+b_{11}\frac{\partial^2 \phi}{\partial x^2}-b_{12}\phi = -b_{10}\frac{\partial^2 \psi}{\partial t^2}+b_{13}\frac{\partial^2 \phi}{\partial t^2} \tag{18}$$

in $0 < x < l$. A system of natural boundary conditions at each end of the beam is also derived as

$$b_1\frac{\partial w}{\partial x}-b_2\psi-b_3\phi = Q/\xi \tag{19}$$

$$b_5\frac{\partial \psi}{\partial x}-b_7\frac{\partial \phi}{\partial x} = \overline{M}/\xi \tag{20}$$

$$b_{11}\frac{\partial \phi}{\partial x}-b_7\frac{\partial \psi}{\partial x} = \overline{m}/\xi. \tag{21}$$

In equations (16)–(21), the coefficients are given by

$$b_1 = \kappa_1 A_1 G_1+\kappa_2 A_2 G_2+A_1\sigma_1^0+A_2\sigma_2^0, \quad b_2 = \kappa_2 A_2 G_2/(1-\eta) \tag{22}$$

$$b_3 = \kappa_1 A_1 G_1-\kappa_2\eta A_2 G_2/(1-\eta), \quad b_4 = \rho_1 A_1+\rho_2 A_2 \tag{23}$$

$$b_5 = (E_2 I_2+\beta I_2\sigma_2^0)/(1-\eta)^2+I_b[\eta E_1+(1-\eta)E_2+\beta\eta\sigma_1^0+\beta(1-\eta)\sigma_2^0]/\xi \tag{24}$$

$$b_6 = b_2/(1-\eta), \quad b_7 = \eta(E_2 I_2+\beta I_2\sigma_2^0)/(1-\eta)^2 \tag{25}$$

$$b_8 = \eta b_6, \quad b_9 = \rho_2 I_2/(1-\eta)^2+\eta\rho_1 I_b/\xi+(1-\eta)\rho_2 I_b/\xi \tag{26}$$

$$b_{10} = \eta\rho_2 I_2/(1-\eta)^2 \tag{27}$$

$$b_{11} = E_1 I_1+\eta^2 E_2 I_2/(1-\eta)^2+\beta I_1\sigma_1^0+\beta\eta^2 I_2\sigma_2^0/(1-\eta)^2 \tag{28}$$

$$b_{12} = \kappa_1 A_1 G_1+\kappa_2\eta^2 A_2 G_2/(1-\eta)^2 \tag{29}$$

$$b_{13} = \rho_1 I_1+\eta^2\rho_2 I_2/(1-\eta)^2. \tag{30}$$

4. APPROXIMATE EQUATIONS

It has been shown in [1] that for a composite beam with no initial stress both the gross and the micro-rotatory inertias can be neglected for a substantial range of longer wavelengths without resulting in significant errors. Thus, we may write an approximate expression for the kinetic energy as

$$T = \frac{1}{2}\xi(\rho_1 A_1+\rho_2 A_2)\left(\frac{\partial w}{\partial t}\right)^2. \tag{31}$$

For most of the structural materials, the assumption, $E_1, G_1 \gg |\sigma_1^0|$ and $E_2, G_2 \gg |\sigma_2^0|$, is often made to avoid yielding. With the foregoing assumption, it is easy to see that the incremental strain energy due to the initial stress associated with the terms $(\partial\psi/\partial x)^2$ and $(\partial\phi_i^k/\partial x)^2$ [see equation (5)] may be neglected, which can be achieved by simply setting $\beta = 0$.

Another consideration leading to a further simplification of the incremental strain energy of the composite beam is the fact that there is usually a substantial difference in rigidity between the two constituent materials. In the range of the wavelengths that are comparable to or smaller than the thickness of beam but are still large as compared with the thickness of the reinforcing layer, the shear deformation of the composite beam is essentially taken by the soft matrix layers. Accordingly, we may neglect the shear deformation, β_1, in the reinforcing layers and write

$$\frac{\partial w}{\partial x} = \phi_1 \equiv \phi. \tag{32}$$

Using equation (32), we can eliminate ϕ from the set of independent kinematic variables, retaining now only w and ψ.

Based upon the foregoing assumptions, we now substitute $\beta = 0$ and equation (32) into equation (11) to obtain an approximate incremental strain energy in the form

$$\begin{aligned}\Delta U = {} & \frac{1}{2}E_1 I_1 \xi \left(\frac{\partial^2 w}{\partial x^2}\right)^2 + \frac{1}{2}E_2 I_2 \xi \frac{1}{(1-\eta)^2}\left(\frac{\partial\psi}{\partial x} - \eta\frac{\partial^2 w}{\partial x^2}\right)^2 \\ & + \frac{1}{2}\kappa_2 \xi A_2 G_2 \frac{1}{(1-\eta)^2}\left(\frac{\partial w}{\partial x} - \psi\right)^2 \\ & + \frac{1}{2}I_b[\eta E_1 + (1-\eta)E_2]\left(\frac{\partial\psi}{\partial x}\right)^2 \\ & + \frac{1}{2}\xi(A_1\sigma_1^0 + A_2\sigma_2^0)\left(\frac{\partial w}{\partial x}\right)^2.\end{aligned} \tag{33}$$

With equations (31), (33) and the equation derived by eliminating ϕ from equation (15) through the use of equation (32), we can, again, invoke Hamilton's principle to derive the approximate equations of motion. We obtain

$$c_1\frac{\partial^4 w}{\partial x^4} - c_2\frac{\partial^2 w}{\partial x^2} - c_3\frac{\partial^3 w}{\partial x^3} + c_4\frac{\partial\psi}{\partial x} + c_5\frac{\partial^2 w}{\partial t^2} = q/\xi \tag{34}$$

$$c_3\frac{\partial^3 w}{\partial x^3} - c_4\frac{\partial w}{\partial x} - c_6\frac{\partial^2\psi}{\partial x^2} + c_4\psi = 0. \tag{35}$$

The corresponding boundary conditions can be expressed in the form

$$-c_1\frac{\partial^3 w}{\partial x^3} + c_2\frac{\partial w}{\partial x} + c_3\frac{\partial^2\psi}{\partial x^2} - c_4\psi = Q/\xi \tag{36}$$

$$c_6\frac{\partial\psi}{\partial x} - c_3\frac{\partial^2 w}{\partial x^2} = \overline{M}/\xi \tag{37}$$

$$c_1\frac{\partial^2 w}{\partial x^2} - c_3\frac{\partial\psi}{\partial x} = \overline{m}/\xi. \tag{38}$$

In equations (34)–(38), the coefficients are given by

$$c_1 = E_1 I_1 + E_2 I_2 \eta^2/(1-\eta)^2 \tag{39}$$

$$c_2 = \kappa_2 A_2 G_2/(1-\eta)^2 + A_1 \sigma_1^0 + A_2 \sigma_2^0 \tag{40}$$

$$c_3 = \eta E_2 I_2/(1-\eta)^2 \tag{41}$$

$$c_4 = \kappa_2 A_2 G_2/(1-\eta)^2 \tag{42}$$

$$c_5 = \rho_1 A_1 + \rho_2 A_2 \tag{43}$$

$$c_6 = E_2 I_2/(1-\eta)^2 + \eta E_1 I_b/\xi + (1-\eta) E_2 I_b/\xi. \tag{44}$$

5. LAW OF MIXTURES

A very widely used approach in evaluating the gross elastic property of a composite material is the law of mixtures, through which an effective modulus of the composite is expressed as the sum of the elastic moduli of the individual constituent materials weighted by the respective volume fractions. In the present case, the effective Young's modulus and the effective shear modulus are obtained as

$$E_v = \eta E_1 + (1-\eta) E_2 \tag{45}$$

$$G_v = \eta G_1 + (1-\eta) G_2, \tag{46}$$

respectively. Taking the effective mass density and the effective initial stress to be

$$\rho_v = \eta \rho_1 + (1-\eta) \rho_2 \tag{47}$$

$$\sigma_v^0 = \eta \sigma_1^0 + (1-\eta) \sigma_2^0, \tag{48}$$

respectively, we can then regard the composite beam as a homogeneous beam. The equations of motion for a homogeneous Timoshenko beam under initial stress derived based on the Trefftz's theory are presented in the Appendix.

6. INFLUENCE OF INITIAL STRESS ON WAVE PROPAGATION

Initial stress has significant effect on oscillations and wave propagation [5, 6]. In this section, harmonic flexural wave propagation in a composite beam under axial compression will be investigated.

We assume that the beam is free from lateral loads, i.e. $q = 0$. According to the microstructure beam theory as developed in Sections 2, 3, the flexural wave propagating in the x-direction is expressed in the form

$$w = hW \exp ik(x-ct) \tag{49}$$

$$\psi = \Psi \exp ik(x-ct) \tag{50}$$

$$\phi = \Phi \exp ik(x-ct) \tag{51}$$

where k is the angular wave number, c the phase velocity and W, Ψ and Φ are constants.

Substitution of equations (49)–(51) in the equations of motion, equations (16)–(18), leads to three equations which can be written in dimensionless form as

$$(B_1 V^2 + B_2)W + B_3 i\Psi + B_4 i\Phi = 0 \tag{52}$$

$$B_5 W + (B_6 V^2 + B_7) i\Psi + (B_8 V^2 + B_9) i\Phi = 0 \tag{53}$$

$$B_{10} W + (B_{11} V^2 + B_{12}) i\Psi + (B_{13} V^2 + B_{14}) i\Phi = 0 \tag{54}$$

where

$$V = c/(G_2/\rho_2)^{\frac{1}{2}} \tag{55}$$

is the dimensionless phase velocity. The coefficients $B_n (n = 1\text{–}14)$ in equations (52)–(54) are given by

$$B_1 = (\theta n_1 + n_2)K^2 \tag{56}$$

$$B_2 = -(\kappa_1 n_1 \gamma + \kappa_2 n_2 - n_1 \gamma p_1 - n_2 p_2)K^2 \tag{57}$$

$$B_3 = -\kappa_2 n_2 K/(1-\eta) \tag{58}$$

$$B_4 = -[\kappa_1 n_1 \gamma - \kappa_2 \eta n_2/(1-\eta)]K \tag{59}$$

$$B_5 = -B_3 \tag{60}$$

$$B_6 = -[\varepsilon_2/(1-\eta)^2 + \eta\theta\varepsilon_b/\xi + (1-\eta)\varepsilon_b/\xi]K^2 \tag{61}$$

$$B_7 = \{(\delta_2\varepsilon_2 - \beta\varepsilon_2 p_2)/(1-\eta)^2 + \varepsilon_b[\eta\delta_1 + (1-\eta)\delta_2 - \beta\eta\gamma p_1 - \beta(1-\eta)p_2/\xi\}K^2 + \kappa_2 n_2/(1-n)^2 \tag{62}$$

$$B_8 = \eta\varepsilon_2 K^2/(1-\eta)^2 \tag{63}$$

$$B_9 = -(\delta_2\varepsilon_2 - \beta\varepsilon_2 p_2)K^2/(1-\eta)^2 - \kappa_2 n_2 \eta/(1-\eta)^2 \tag{64}$$

$$B_{10} = -B_4 \tag{65}$$

$$B_{11} = B_8 \tag{66}$$

$$B_{12} = B_9 \tag{67}$$

$$B_{13} = -[\theta\varepsilon_1 + \varepsilon_2\eta^2/(1-\eta)^2]K^2 \tag{68}$$

$$B_{14} = [\delta_1\varepsilon_1 - \beta\varepsilon_1\gamma p_1 + \eta^2(\delta_2\varepsilon_2 - \beta\varepsilon_2 p_2)/(1-\eta)^2]K^2 + \kappa_1 n_1 \gamma + \kappa_2 \eta^2 n_2/(1-\eta)^2 \tag{69}$$

where

$$K = hk, \quad \gamma = G_1/G_2, \quad \theta = \rho_1/\rho_2 \tag{70}$$

$$n_1 = A_1/h^2, \quad n_2 = A_2/h^2, \quad \delta_1 = E_1/G_2, \quad \delta_2 = E_2/G_2 \tag{71}$$

$$\varepsilon_1 = I_1/h^4, \quad \varepsilon_2 = I_2/h^4, \quad \varepsilon_b = I_b/h^4 \tag{72}$$

$$p_1 = -\sigma_1^0/G_1, \quad p_2 = -\sigma_2^0/G_2. \tag{73}$$

The dispersion equation can now be obtained by setting the determinant of coefficients of equations (52)–(54) to zero. The determinantal equation can be expressed as a cubic equation for V^2. To exhibit the influence of the initial stress on the phase velocity we now solve the dispersion equation numerically for $\eta = 0{\cdot}8$ and the following values:

$$G_1/G_2 = 100, \quad \rho_1/\rho_2 = 2, \quad \xi = 4{\cdot}8, \quad E_1/G_2 = 240, \quad E_2/G_2 = 2{\cdot}7. \tag{74}$$

In the computation, we also assume that $b = h$. Thus, we have

$$\begin{aligned} n_1 &= \eta/\xi, \quad n_2 = (1-\eta)/\xi, \quad \varepsilon_1 = \eta^2/12\xi^3, \\ \varepsilon_2 &= (1-\eta)^3/12\xi^3, \quad \varepsilon_b = 1/12. \end{aligned} \tag{75}$$

The dispersion curves for the lowest mode for the two different initial stresses as represented by $p = p_1(= -\sigma_1^0/G_1) = p_2(= -\sigma_2^0/G_2) = 0{\cdot}01$ and $p = p_1 = p_2 = 0{\cdot}1$ in conjunction with the parameters as given by equations (74) and (75) are shown in Fig. 2 with V vs. h/λ, where

$$\lambda = 2\pi/k = 2\pi h/K \tag{76}$$

is the wavelength. As may have been expected, the compressive initial stress lowers the phase velocity by a significant amount. The earlier surmise that the terms attached by the coefficient $\beta(= 1, 0)$ may be neglected when the initial stress is small is now confirmed at least for the cases considered here.

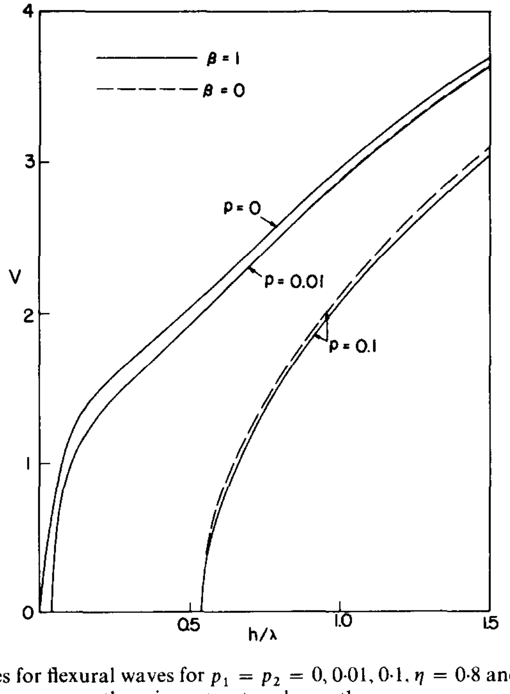

FIG. 2. Dispersion curves for flexural waves for $p_1 = p_2 = 0, 0{\cdot}01, 0{\cdot}1$, $\eta = 0{\cdot}8$ and $\beta = 1, 0$ according to the micro-structure beam theory.

According to the approximate theory as developed in Section 4, the flexural wave is described by equations (49) and (50). The dispersion equation is derived following the previous procedure. We obtain

$$V = \left\{ \frac{P_3 P_4 - P_2 P_5}{P_1 P_5} \right\}^{1/2} \tag{77}$$

where

$$P_1 = (\theta n_1 + n_2)K \tag{78}$$

$$P_2 = -[\delta_1 \varepsilon_1 + \delta_2 \varepsilon_2 \eta^2/(1-\eta)^2]K^3 - [\kappa_2 n_2/(1-\eta)^2 - n_1 \gamma p_1 - n_2 p_2]K \tag{79}$$

$$P_3 = -(\delta_2 \varepsilon_2 \eta K^2 + \kappa_2 n_2)/(1-\eta)^2 \tag{80}$$

$$P_4 = -P_3 K \tag{81}$$

$$P_5 = -[\delta_2 \varepsilon_2/(1-\eta)^2 + \eta \delta_1 \varepsilon_b/\xi + (1-\eta)\delta_2 \varepsilon_b/\xi]K^2 - \kappa_2 n_2/(1-\eta)^2. \tag{82}$$

For $p = p_1 = p_2 = 0{\cdot}01$ and $\eta = 0{\cdot}5$, $0{\cdot}8$ together with the numerical values as given by equations (74)–(75), the dimensionless phase velocity is plotted according to equation (77) in Fig. 3. The corresponding dispersion curves obtained according to the micro-structure beam theory (with $\beta = 1$) and the law of mixtures are also shown. The curves based on the approximate equations show both qualitative and quantitative agreement with the "exact"

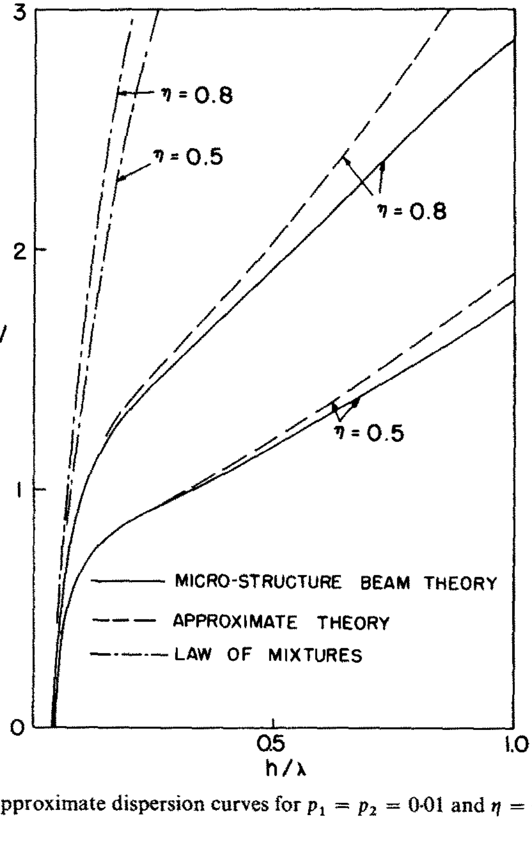

FIG. 3. Approximate dispersion curves for $p_1 = p_2 = 0{\cdot}01$ and $\eta = 0{\cdot}8, 0{\cdot}5$.

curves over a substantial range of wavelengths. The dispersion curves calculated according to the law of mixtures, however, deviate substantially from the others. Similar results also occurred in the case where no initial stress was present [1]. The discrepancy can be resorted to the fact that the micro-shear deformation in the layers, which the law of mixtures fails to account for, becomes pronounced as the wavelength decreases.

7. STABILITY

It is well known that the phase velocity of the flexural wave in a beam decreases with increasing compressive axial stress, and approaches to zero as the beam buckles. Thus, by setting $V = 0$ in the dispersion equations as obtained in the previous section, we can obtain the equations for the buckling loads based on the three theories.

We again assume that $p = p_1 = p_2$. The critical buckling stress parameter will be designated by p_{cr}. According to the micro-structure beam theory, the equation that p_{cr} has to satisfy is derived from the dispersion equation by setting $V = 0$. The thus obtained equation is solved numerically for $\eta = 0{\cdot}8$ and $\eta = 0{\cdot}5$, together with the values as given by equations (74), (75). The results are plotted in Fig. 4 with p_{cr} vs. h/λ for the two cases $\beta = 1$ and $\beta = 0$. It is noted that the difference between these two cases is negligible in the range of wavelengths where the buckling stresses are small as compared with the shear moduli, i.e. $G_1 \gg |\sigma^0_{1\,cr}|$ and $G_2 \gg |\sigma^0_{2\,cr}|$ ($\sigma^0_{1\,cr}$ and $\sigma^0_{2\,cr}$ are the critical stresses in the reinforcing and the matrix layers, respectively).

The approximate theory as discussed in Section 4 can also be employed to predict the buckling load. Setting $V = 0$ in equation (77), we obtain

$$p_{cr} = \{P_3P_4/P_5K + [\delta_1\varepsilon_1 + \delta_2\varepsilon_2\eta^2/(1-\eta)^2]K^2 + \kappa_2 n_2/(1-\eta)^2\}/(n_1\gamma + n_2) \qquad (83)$$

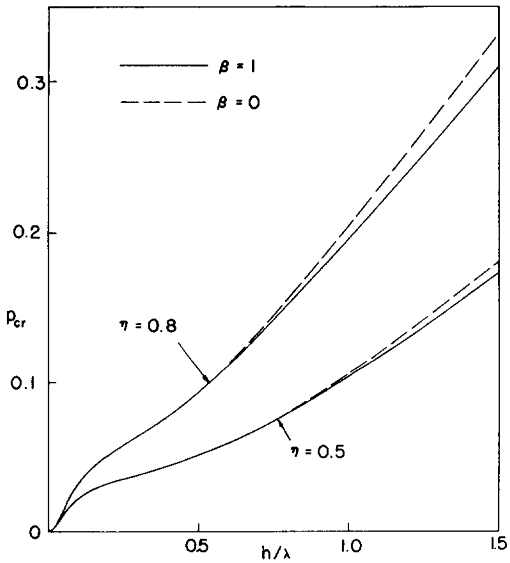

FIG. 4. Buckling of composite beam with p_{cr} as a function of h/λ for $\eta = 0{\cdot}8, 0{\cdot}5$, $\beta = 1, 0$ according to the micro-structure beam theory.

where P_3, P_4 and P_5 are given by equations (80)–(82). In an analogous manner, the buckling stress parameter can also be calculated based on the law of mixtures.

In Fig. 5, p_{cr} is plotted vs. h/λ according to the foregoing three theories for $\eta = 0{\cdot}8, 0{\cdot}5$ and the parameters as given by equations (74), (75). It is seen that the approximate theory agrees with the micro-structure theory for a wide range of wavelengths. The agreement is better when η assumes a smaller value, that is, when the matrix layer is thicker than the reinforcing layer. The reason is that for such composite beams the matrix layers can take more shear deformations, and, as a consequence, the assumption concerning the shear deformation made in the approximate theory becomes more adequate. On the other hand, the result according to the law of mixtures departs from the other two theories at very large wavelengths. It is observed that the buckling loads predicted by the law of mixtures

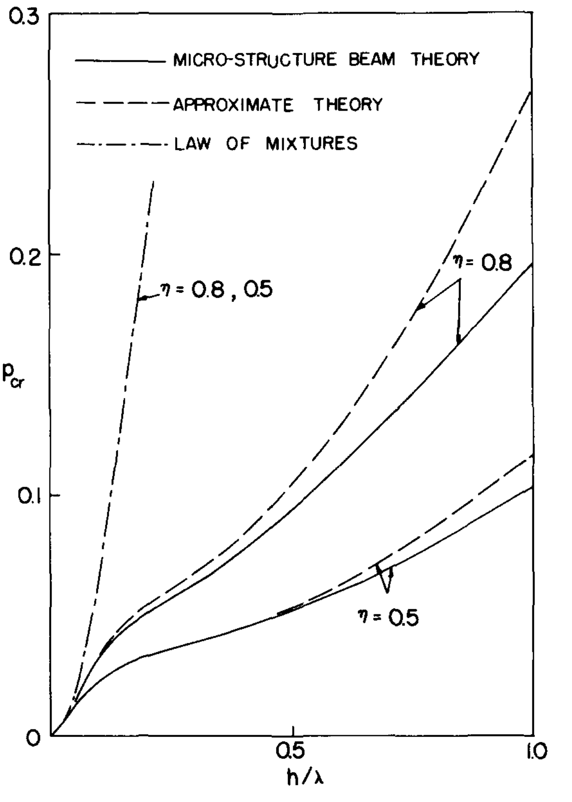

FIG. 5. Buckling of composite beam for $\eta = 0{\cdot}8, 0{\cdot}5$, $\beta = 1$ according to the micro-structure beam theory, the approximate theory and the law of mixtures.

are much larger. It is interesting to note that, in contrast with the micro-structure beam theory, the ratio of the thicknesses of the reinforcing and matrix layers have little influence on the buckling stress according to the law of mixtures.

As indicated by Fig. 5, the buckling stress increases as the wavelength decreases. In the range of shorter wavelengths that are comparable to the thicknesses of the layers, internal buckling of the constituent layers may occur [7, 8] and the structural buckling as has been discussed in this paper may not be the dominant factor in the consideration of the compression strength of the composite beam. Moreover, at high stress level, most composites would undergo plastic deformation and failure due to kinking of fibers and the delamination of the layers would become likely [9].

Acknowledgements—The author wishes to thank Professor J. D. Achenbach of Northwestern University for helpful discussions. This work was supported by a grant from the National Science Foundation.

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APPENDIX

Equations of motion for a Timoshenko beam under initial stress

Consider a straight beam of cross-sectional area A which is subjected to a uniform initial stress σ^0 applied along the axis of the beam (x-axis). We assume that the incremental displacements in the beam can be approximated by

$$\bar{u}(x, y, t) = u(x, t) - y\phi(x, t) \tag{A1}$$

$$\bar{w}(x, y, t) = w(x, t) \tag{A2}$$

where $\bar{u}$ and $\bar{w}$ are the displacements in the x- and y-directions, respectively. The pertinent Lagrangian strain components according to equations (A1) and (A2) are

$$e_{xx} = \frac{\partial u}{\partial x} - y\frac{\partial \phi}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x} - y\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{A3}$$

$$e_{xy} = \frac{1}{2}\left(\frac{\partial w}{\partial x} - \phi\right) - \frac{1}{2}\phi\left(\frac{\partial u}{\partial x} - y\frac{\partial \phi}{\partial x}\right). \tag{A4}$$

Let σ_{xx} and σ_{xy} be the pertinent Trefftz's incremental stresses which are taken as related to the accompanying deformation by linear relations of the type

$$\sigma_{xx} = Ee_{xx} \tag{A5}$$

$$\sigma_{xy} = 2\kappa Ge_{xy} \tag{A6}$$

where κ is the shear correction coefficient whose value has been determined in [2] to be $\pi^2/12$. In equations (A5) and (A6), it is noted that the values of the elastic moduli E and G depend, in general, on the value of the initial stress. With the linear stress–strain relations, the strain energy potential per unit initial volume can be expressed in the form

$$W = (\sigma^0 + \tfrac{1}{2}\sigma_{xx})e_{xx} + \sigma_{xy}e_{xy}. \tag{A7}$$

Substituting equations (A5) and (A6) in equation (A7) and subsequently integrating over the thickness of the beam, we obtain the total strain potential energy per unit initial length of the beam as

$$\begin{aligned} U = {} & A\sigma^0\frac{\partial u}{\partial x} + \frac{1}{2}A\sigma^0\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}I\sigma^0\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}A\sigma^0\left(\frac{\partial w}{\partial x}\right)^2 \\ & + \frac{1}{2}EA\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}EI\left(\frac{\partial \phi}{\partial x}\right)^2 + \frac{1}{2}\kappa GA\left(\frac{\partial w}{\partial x} - \phi\right)^2 \end{aligned} \tag{A8}$$

where I is the area moment of inertia of the cross-section. It is noted that in deriving equation (A8) the third order terms have been neglected.

Taking into account the condition that the initial stresses and initial boundary forces are in a state of equilibrium, we note that only the incremental strain energy,

$$\Delta U = U - A\sigma^0\frac{\partial u}{\partial x}, \tag{A9}$$

is pertinent in the variational formulation [4].

The kinetic energy per unit length of the beam is

$$T = \frac{1}{2}\rho A\left(\frac{\partial w}{\partial t}\right)^2 + \frac{1}{2}\rho I\left(\frac{\partial \phi}{\partial t}\right)^2 + \frac{1}{2}\rho A\left(\frac{\partial u}{\partial t}\right)^2. \tag{A10}$$

The equations of motion can be obtained by applying Hamilton's principle. If the beam is free from lateral loads, then the equations for flexural motions ($u = 0$) are obtained as

$$(\sigma^0 + \kappa G)\frac{\partial^2 w}{\partial x^2} - \kappa G\frac{\partial \phi}{\partial x} = \rho\frac{\partial^2 w}{\partial t^2} \tag{A11}$$

$$I(\sigma^0 + E)\frac{\partial^2 \phi}{\partial x^2} + \kappa GA\left(\frac{\partial w}{\partial x} - \phi\right) = \rho I\frac{\partial^2 \phi}{\partial t^2}. \tag{A12}$$

(*Received* 1 *March* 1971; *revised* 14 *May* 1971)

Абстракт – Исследуется составная балка, состоящая из большого числа слоев, изготовленных из разных упругих материалов, под влиянием начальных напряжений. Принимается, что составлящие слоя ведут себя как балки Тимошенки с начальными напряжениями. С помощью несложной операции преобразовуется составная балка в макрооднородную балку с микроструктурой. Путем принципа Гамильтона оиределяются уравнения движения и граничные условия. Дается система приближенных уравнений, основанная в части на заметно разных жесткостях двух составляющих материалов. Исследуются распространения волн изгиба и устойчивость составной балки, используя теорию балки с микроструктуркой, приближенную теорию и закон смесей. Даются числовые примеры.